

1980-3

A THREE-DIMENSIONAL MATHEMATICAL MODEL OF THE KNEE-JOINT*

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Abstract - A three-dimensional analytical model of the knee-joint is presented, taking into account the geometry of the joint surfaces as well as the geometry and material properties of the ligaments and capsule. The position of a large number of points on the joint surfaces is measured and the geometry of these surfaces is then approximated by polynomials in space. The ligaments and capsule are represented by a number of non-linear springs, with material properties selected from the literature. For a given three-dimensional loading (forces as well as moments) at various flexion-extension angles, the location of contact points, magnitude and direction of contact forces, magnitude of ligament elongation and ligament forces can be calculated.

In the results presented in this paper special attention is given to the anterior-posterior laxity of a joint. A sensitivity study was undertaken to evaluate the model response due to some of the model parameters and to gain a better understanding of the function of the elements in the model. It is concluded that the predictions of the model agree well with experiments described in the literature.

INTRODUCTION

Mathematical models offer expanding possibilities for the analysis of complicated biological structures. This paper presents a model for the analysis of the motions and forces between two body segments. The human knee joint has been selected for this study as it has a complicated anatomical structure and a complicated three-dimensional motion. Not only a faithful description of normal function, but also identification and treatment of disfunction presents many problems.

Kinematical models of the knee joint, based on the theory of the four-bar mechanism, have been developed by Zuppinger (1904), Menschik (1974) and Huson (1974). In this type of model force action in the structures of the joint is not considered.

In the models developed by Morrison (1970) and Crowninshield (1976) force action in these structures is studied but several simplifications were introduced concerning the kinematical behaviour. In the model of Morrison (1970) the knee joint was assumed to be a simple hinge joint and in the model of Crowninshield (1976) the motions in the joint were based on experimental data in the literature, which are, however, often

contradictory. Moreover, the contribution of the curved joint surfaces to the mechanical behaviour was ignored in these models.

Recently, Andriacchi (1977) reported the development of a model for the analysis of the motions and forces in the knee joint, employing finite element methods. Ligaments and capsule were represented by non-linear springs, while the joint surfaces were modelled by a number of flat surfaces.

The model presented here takes into account the ligaments and capsule and the three-dimensional geometry of the joint surfaces. The curved joint surfaces are represented by polynomials in space. Since three-dimensional geometrical data of the joint surfaces are not available in the literature, these data were measured on anatomical specimens. The model enables the following to be calculated as a function of flexion-extension angle and external forces: the position of the femur relative to the tibia, ligament forces, ligament elongations, position of contact points and magnitude and direction of contact forces.

This paper reports the theoretical background of the model. Some results of calculations are also presented to demonstrate the possibilities the model offers and to show the agreement with experimental results. Special attention is given to the anterior-posterior laxity of the knee-joint.

FORMULATION OF THE MODEL

Assumptions and simplifications

This study is limited to the quasi-static behaviour of the femoro-tibial joint. However, the patello-femoral

*Received 12 April 1979; in revised form 6 November 1979.

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joint can be simulated in an equivalent way. The model includes a representation of the proximal portion of the tibia, the distal portion of the femur and the structures that connect femur to tibia (ligaments and joint capsule). Like experiments described in the literature (eg Brantigan, 1941 and Markolf, 1976), the model describes the relative position of the femur to the tibia as a function of the flexion-extension angle and an external load.

Deformations of the bones and of the cartilage layer of the condyles are ignored as these are relatively small compared to the displacements in the joint (Walker, 1972).

The influence of menisci on the relative motion has been studied by several authors (eg Markolf, 1976). Although it was found that in some cases the menisci influence the so-called stability of a joint, the menisci are neglected in this model: introducing the menisci in a realistic way would complicate the model considerably.

As a consequence of these assumptions, the outer surface of each of the condyles of femur and tibia can be simulated by a curved rigid surface, while the contact areas between femur and tibia are reduced to contact points.

Friction between femoral and tibial joint surfaces will be ignored since the coefficient of friction between cartilage surfaces, owing to the synovial fluid, is very low (Radin, 1972).

The relative position of femur to tibia

To describe the relative position of the femur to the tibia, the tibia can be considered to be rigidly fixed. The geometrical data of the tibia are described in a fixed orthogonal co-ordinate system (x, y, z) with unit vectors e_x , e_y and e_z (Fig. 1). The axis of the tibia coincides with the y -axis, the x -axis is oriented in posterior direction and the z -axis in lateral direction. The geometrical data of the femur are described in an orthogonal co-ordinate system (α, β, γ) with unit vectors e_α , e_β and e_γ . This system is fixed to the femur. The axis of the femur coincides with the β -axis.

If the position of an arbitrary point on the femur is indicated by the vector δ in the (α, β, γ) -system and by the vector c in the (x, y, z) -system then:

$$c = a + T\delta \quad (1)$$

where a is the vector from the origin of the (x, y, z) -system to the origin of the (α, β, γ) -system and T is a (3×3) orthogonal rotation matrix. T is considered to be the result of three subsequent rotations ψ , ω and ϕ :

$$T = T(\psi, \omega, \phi) \quad (2)$$

where ϕ is the so-called flexion-extension angle, defined by the angle between the y -axis and the projection of the β -axis on the (x, y) -plane (in extension $\phi = 0$). For a detailed description of ϕ , ψ and ω see Wismans (1980).

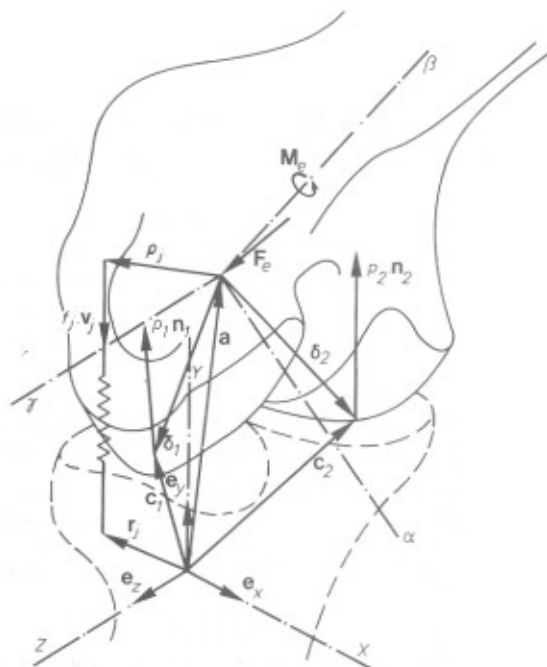


Fig. 1. Model of the knee-joint.

Mathematical description of the joint surfaces

Since no realistic data for the geometry of the outer surfaces of the condyles could be derived from the literature, a device was developed for measuring these data on an anatomical specimen (Fig. 2). With this device the femur and the tibia are measured separately. It consists of a dial gauge, which can move relative to the femur (tibia) in two perpendicular directions. The three co-ordinates of the dial gauge end placed on the joint surface are recorded. A computer program has been developed to correct for the radius ($= 1$ mm) of the dial gauge end. In this way a number of points (50–100) on each condyle are measured. Deformations of the cartilage layer, caused by the dial gauge end, are ignored, as these are relatively small (< 0.1 mm). Femur and tibia are provided with reference points, with a known relative position in the extension position of the intact point. By measuring the co-ordinates of these points, the original extension position can be reconstructed later on.

The position vector c_i of a point on the relevant part of the outer surface of condyle i of the tibia ($i = 1$ for the lateral condyle and $i = 2$ for the medial one) is given by

$$c_i = x e_x + y_i(x, z) e_y + z e_z \quad (3)$$

where x , $y = y_i(x, z)$ and z are the co-ordinates of that point and y is considered as a function of x and z . This function is approximated by a polynomial in x and z and of degree n :

$$y_i(x, z) = \sum_{i=0}^n \sum_{j=0}^{n-i} a_{ij} x^i z^j \quad (4)$$

The coefficients a_{ij} are calculated by minimizing the function

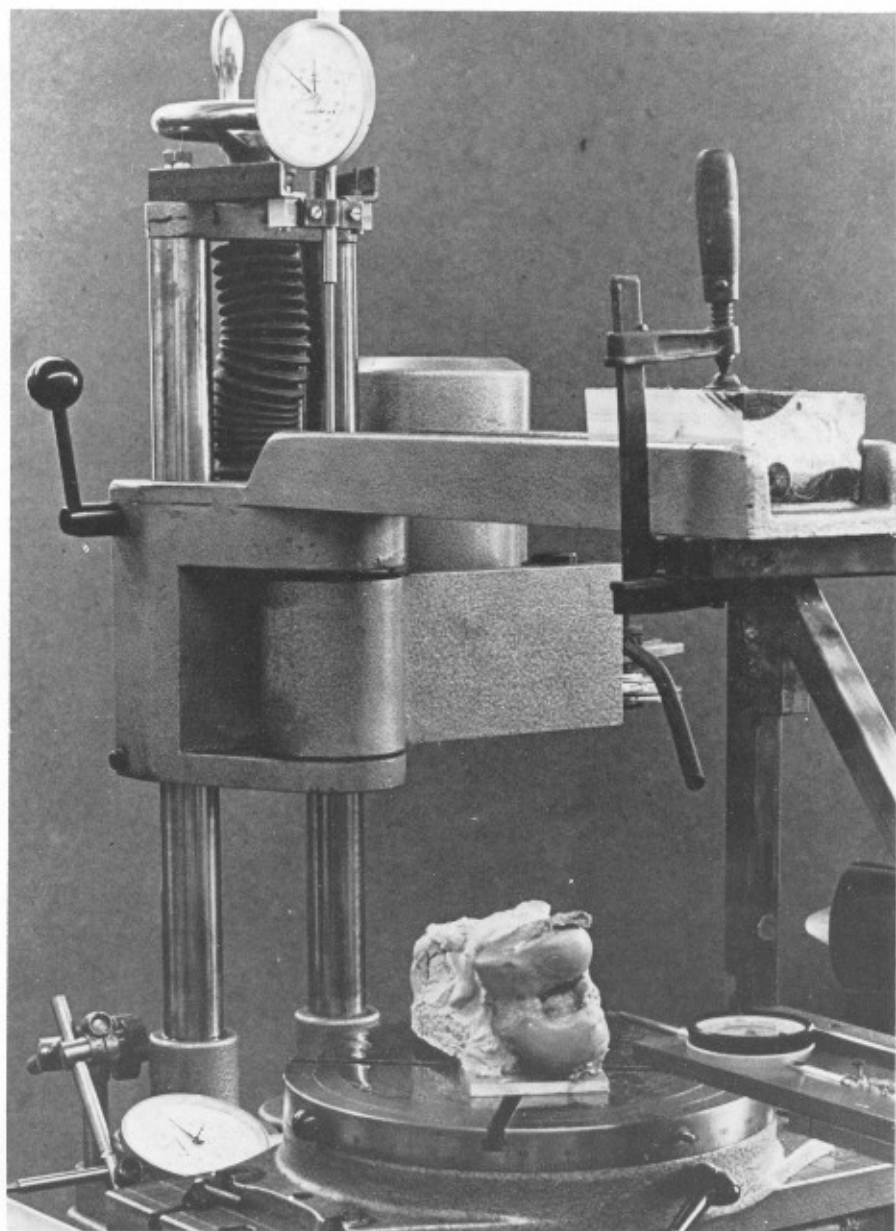


Fig. 2. Device for measuring the geometry of a joint.

$$\sum_{r=1}^m \{ \tilde{y}_r - y_i(\tilde{x}_r, \tilde{z}_r) \}^2 = \sum_{r=1}^m \left\{ \tilde{y}_r - \sum_{i=0}^n \sum_{j=0}^{n-i} a_{ij} \tilde{x}_r^i \tilde{z}_r^j \right\}^2 \quad (5)$$

Here, m is the number of measured points and $(\tilde{x}_r, \tilde{y}_r, \tilde{z}_r)$ are the measured co-ordinates of point r . In the same way the position vector δ_i of a point on the relevant part of the outer surface of condyle i of the femur is approximated by

$$\delta_i = \alpha \mathbf{e}_x + \beta_i(\alpha, \gamma) \mathbf{e}_\beta + \gamma \mathbf{e}_\gamma \quad (6)$$

where $\alpha, \beta = \beta_i(\alpha, \gamma)$ and γ are the co-ordinates of that point and $\beta_i = \beta_i(\alpha, \gamma)$ is a polynomial in α and γ .

The accuracy of the approximation will depend on the degree of the polynomial. In Fig. 3, which shows the measured parts of the joint surfaces and a computer representation of the mathematical approximations, the degree is 3 for the tibia and 4 for the femur. For this case the standard deviation is smaller than 0.5 mm.

Contact between femur and tibia

In the model contact between femur and tibia at both condyles $i = 1$ and $i = 2$ is required. As a consequence, varus-valgus motions cannot be studied with the model. For each point of contact the following equations must hold:

$$\mathbf{c}_i = \mathbf{a} + T\delta_i \quad (7)$$

$$(\mathbf{n}_i, T\boldsymbol{\tau}_{xi}) = 0 \quad (8)$$

$$(\mathbf{n}_i, T\boldsymbol{\tau}_{yi}) = 0 \quad (9)$$

where \mathbf{n}_i is a unit outward normal vector of the tibia condyle i and

$$\boldsymbol{\tau}_{xi} = \frac{\partial \delta_i}{\partial \alpha}, \quad \boldsymbol{\tau}_{yi} = \frac{\partial \delta_i}{\partial \gamma} \quad (10)$$

are independent tangent vectors to the femoral condyle i . Equation (7) specifies that the contactpoint on the femoral condyle i coincides with the contactpoint on the tibial condyle i , while equations (8) and (9) specify that the normal vector in the tibial contactpoint i is perpendicular to the tangent plane in the contactpoint of the femoral condyle i . These equations,

the so-called contact-conditions, form a set of 10 independent non-linear relations.

Mathematical description of ligaments and capsule

Ligaments and capsule are represented by m springs. The insertion point of spring j on the femur is denoted by a vector ρ_j in the (α, β, γ) -system, while the insertion point of this spring on the tibia is denoted by a vector \mathbf{r}_j in the (x, y, z) -system (see Fig. 1). The positions of these insertion points are measured on the anatomical specimen used for the joint surface measurements. With equation (1) the length l_j of spring j can be determined:

$$l_j = \sqrt{\{[(\mathbf{r}_j - \mathbf{a} - T\rho_j), (\mathbf{r}_j - \mathbf{a} - T\rho_j)]\}} \quad (11)$$

Soft tissues like ligaments and capsule are known to have visco-elastic properties. Because of the lack of accurate descriptions of these properties and for simplicity the springs representing these structures are assumed to be elastic. The mechanical behaviour is approximated by a quadratic force-elongation function:

$$f_j = k_j(l_j - l_{oj})^2 \quad \text{if } l_j > l_{oj} \quad (12)$$

$$f_j = 0 \quad \text{if } l_j \leq l_{oj}$$

where f_j is the force in spring j , k_j is a constant and l_{oj} is the unstrained length of spring j . The constant k_j indicates the stiffness of spring j and its numerical value is based on experimental work of Trent (1976).

The strain ε_j in spring j is defined by:

$$\varepsilon_j = \frac{l_j - l_{oj}}{l_{oj}} \quad (13)$$

Accurate data on the strain in the soft structures of the knee joint as function of the flexion-extension angle are not available. Only a rough indication of this strain can be obtained from the literature (eg Brantigan, 1941). Assumptions based on these rough data are made for the strain in the springs in extension. This strain is called initial strain and is indicated by ε_{rj} for spring j . The unstrained length l_{oj} of spring j can be calculated then with equation (13).

In the simulation of the knee joint, presented at the end of this paper, ligament and capsule are represented by seven springs (see Fig. 4). The constant k_j and the initial strain ε_{rj} for this simulation are presented in Table 1.

Equilibrium of the femur

The forces and the moments on the femur can be divided into two groups:

(a) *Internal loads.* Forces in the springs, representing ligament and capsule as given by equation (12) and the contact forces between femur and tibia:

$$\mathbf{p}_i = p_i \mathbf{n}_i \quad (i = 1, 2). \quad (14)$$

(b) *External loads.* Muscles, inertia, gravity, patella or other external forces represented by a force \mathbf{F}_e working on the femur in the origin of the (α, β, γ) -

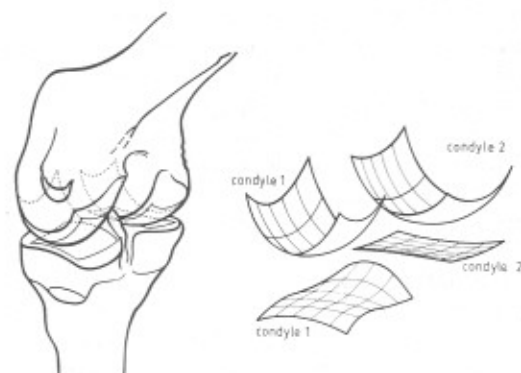


Fig. 3. Measured parts of the joint surfaces and mathematical approximations.

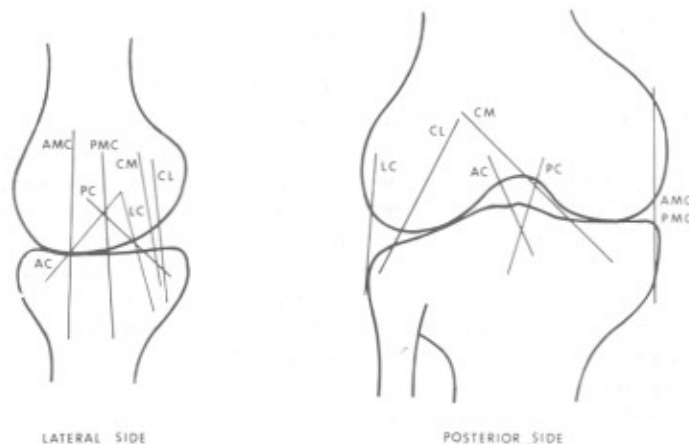


Fig. 4. Position of the springs, representing ligaments and capsule.

system and a moment M_e

$$F_e = F_x e_x + F_y e_y + F_z e_z \quad (15)$$

$$M_e = M_x e_x + M_y e_y + M_z e_z \quad (16)$$

Besides this a moment M_r is working to achieve the prescribed flexion-extension angle ϕ :

$$M_r = M_r \lambda \quad (17)$$

where the vector λ can be calculated employing the principle of virtual work (Wismans, 1980). In general this moment will be zero, except for extreme positions of the joint like hyperextension. The equations for force and moment equilibrium of the femur are used to find the position of the femur relative to the tibia as function of ϕ , F_e and M_e :

$$F_e + p_1 n_1 + p_2 n_2 + \sum_{j=1}^m f_j v_j = 0 \quad (18)$$

$$M_e + M_r \lambda + p_1 (T \delta_1) \times n_1 + p_2 (T \delta_2) \times n_2 + \sum_{j=1}^m \{f_j (T \rho_j) \times v_j\} = 0 \quad (19)$$

where v_j is a unit vector from the femoral to the tibial insertion point.

Equations (18) and (19) are 6 non-linear relations. Together with the contact conditions (7), (8) and (9), the knee-joint is described now by a set of 16 non-linear equations with 16 unknowns, being: the components of the vector a

Table 1. Constant k_j and initial strain ϵ_{rj}

Structure	Abbrev.	k_j (N/mm ²)	ϵ_{rj} (%)
Lateral collateral	LC	15	5
Ant. cruciate	AC	30	5
Post. cruciate	PC	35	-1
Ant. med. collateral	AMC	15	-3
Post. med. collateral	PMC	15	5
Lat. post. capsule	CL	10	5
Med. post. capsule	CM	10	5

the angles ψ and ω

the variables indicating the contact points: x_i , z_i , α_i , γ_i ($i = 1, 2$)

the magnitude of the contact forces: p_1 and p_2

the magnitude of the moment: M_r .

After a reduction of this set, a numerical solution is achieved by employing a Newton-Raphson iteration process.

SOME RESULTS OF THE MODEL

In the calculations presented here the joint surfaces are represented by the curved surfaces as shown in Fig. 3. Ligaments and capsule are represented by seven springs (see Fig. 4). The flexion-extension motion is simulated by prescribing several flexion-extension angles:

$$\phi = 0^\circ, \phi = 5^\circ, \dots, \phi = 100^\circ.$$

In a knee-joint specimen, depending on the magnitude of ϕ a certain amount of back lash can be observed: anterior-posterior laxity, rotatory laxity and varus-valgus laxity *eg* Wang, 1974, Hsieh, 1976, Markolf, 1976). So, if no external load is prescribed, several equilibrium positions can exist for a specified flexion-extension angle. In this paper special attention is given to the anterior-posterior laxity, which can be studied by prescribing a positive, respectively negative force F_x in x -direction. Figure 5 presents the displacement $V_{x \text{ pos}}$ of the origin of the (α, β, γ) -system in positive x -direction, caused by a force $F_x = +10$ N. Similarly, the displacement $V_{x \text{ neg}}$ caused by a force $F_x = -10$ N is presented. These relatively small forces result in relatively large displacements for ϕ between 15 and 55 degrees. So in this part of the flexion-extension motion the anterior-posterior laxity is rather high.

This laxity decreases if a compressive force is prescribed: *eg* by a force $F_y = -500$ N ($\pm 2/3$ body weight), the anterior-posterior laxity decreases by 80 per cent, which is in agreement with experimental work of Hsieh (1976).

To compare anterior-posterior laxity resulting

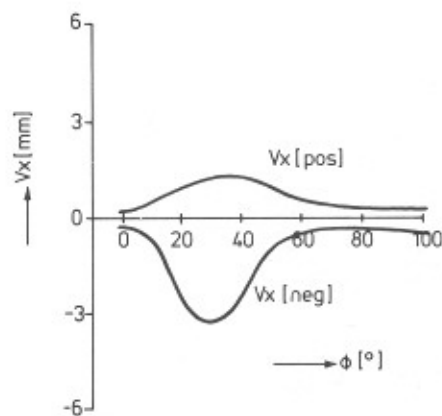


Fig. 5. Displacement of the femur in positive ($V_{x\text{pos}}$) resp. negative ($V_{x\text{neg}}$) x -direction caused by a positive or negative force $F_x = 10\text{ N}$ as function of the flexion-extension angle ϕ .

from the model with experiments of Markolf (1976), calculations were carried out with higher forces F_x . To avoid the problem of defining a neutral joint position, the total displacement $V_x = V_{x\text{pos}} + V_{x\text{neg}}$ was considered. Figure 6 presents for $\phi = 0^\circ$, $\phi = 20^\circ$, $\phi = 50^\circ$ and $\phi = 90^\circ$ the model results together with the results of Markolf. Markolf's data represent an average of 35 specimens; major differences, however, were found in the behaviour of several specimens.

Position of contact points

The location of the contact points as a function of the flexion-extension angle is given in Fig. 7. Results are given for $F_x = +10\text{ N}$ and $F_x = -10\text{ N}$. The contact points on the tibia move in agreement with experimental observations (Walker, 1972), 10–15 mm in posterior direction. The first part of the flexion-extension motion ($\phi: 0^\circ \dots 25^\circ$) is mainly a rolling motion, while in the second part a gliding

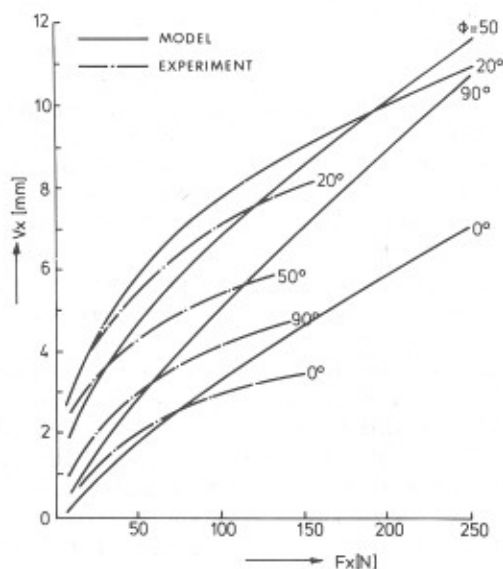


Fig. 6. Displacements V_x of the femur in x -direction as function of an external force F_x at four flexion-extension angles.

motion prevails. These results are in agreement with the work of Zuppinger (1904).

Strain in ligaments and capsule

The strain ϵ_i in the springs as a function of the flexion-extension angle, is given in Fig. 8. Calculations were carried out for $F_x = +10\text{ N}$ and $F_x = -10\text{ N}$. Figure 8 shows the average strain. A negative strain (dotted line) indicates a tensionless state of the ligament or the capsule.

As flexion proceeds, the lateral collateral ligament (LC) and the posterior capsule (CL and CM) decrease in length, which is in agreement with experimental observations.

As each of the cruciates has been simulated by one spring, and as the cruciates show a different tension state in several parts (eg Wang, 1973) it is not possible at this juncture to compare model predictions with literature data. Moreover there are many contradictory statements about the changes in length of the cruciates.

The behaviour of the two parts of the medial collateral ligament (AMC and PMC) is in agreement with Bartel's experiments (1977) on the changes in length of corresponding ligament parts.

Sensitivity analysis

The sensitivity analysis study was undertaken to evaluate the model response due to some of the model parameters and to gain a better understanding of the function of the elements in the model. A complete and detailed documentation of this study is given by Wismans (1978, 1980). In this paper a tabular summary of the influence of some of the parameter variations on the anterior-posterior laxity ($F_x = \pm 10\text{ N}$) at four flexion-extension angles is given (Table 2).

(1) *Stiffness.* Ligament stiffness in the reference run was based on literature data (Trent, 1976), representing an average of 6 specimens. The stiffness of the individual specimens did not exceed $2 \times$ the average values. So a calculation was carried out with the constant k_i doubled. The effect of this variation on the anterior-posterior laxity was found to be rather low.

(2, 3) *Initial strain.* The initial strain of the springs was estimated from very rough descriptions in the literature. The variability of this parameter between specimens is expected to be fairly high. Moreover, this strain will not be constant in a ligament or capsule structure. In the calculations the effect on the anterior-posterior laxity of a relatively small variation in the initial strains was found to be very important.

(4, 5, 6) *Ligament insertions.* The insertion areas of the ligaments and capsule were measured on the anatomical specimen used for the joint surface measurements. The spring insertions were located within these insertion areas. Table 2 records the effect of some variations of the femoral insertion of the spring representing the anterior cruciate. These variations were chosen within the insertion area of the anterior cruciate. A shift of the insertion in lateral

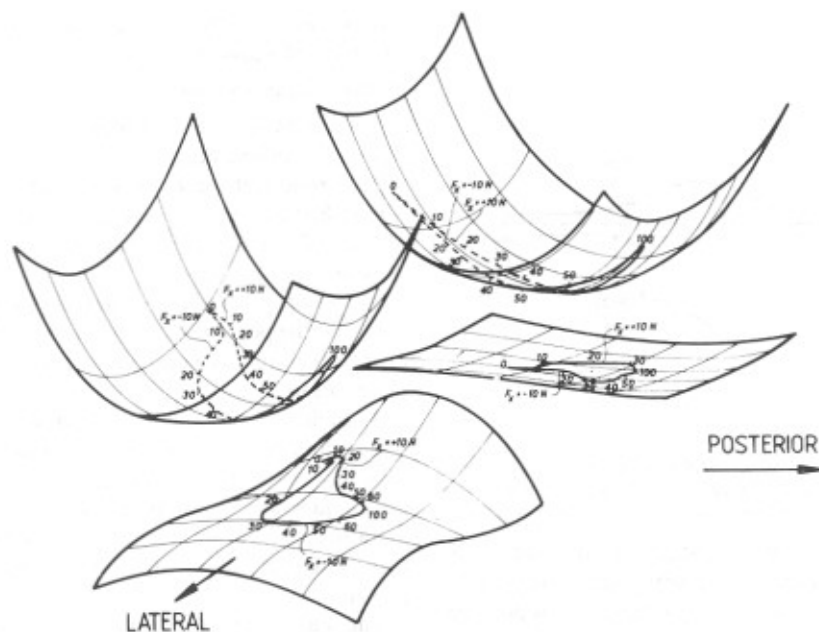


Fig. 7. Position of the calculated contact points as function of the flexion-extension angle ϕ and of an external force $F_x = +10\text{ N}$ and $F_x = -10\text{ N}$.

direction ($z = +5\text{ mm}$) was found to have no significant effect. Displacements, however in a sagittal plane, effect in the flexion position of the joint the anterior-posterior laxity.

(7,...11) *Cut of ligaments.* In experiments with specimens the influence of ligaments were studied by cutting these structures (eg Brantigan, 1941). In the model this can be (simply) simulated by omitting one or more springs from the model. From the results of the calculations presented in Table 2 it can be concluded that the anterior-posterior laxity is mainly affected by the anterior and posterior cruciate, which accords with experimental findings.

DISCUSSION AND CONCLUSIONS

Validation of a model is established when the model predictions correlate acceptably with observed facts. No validation experiments were planned as part of the

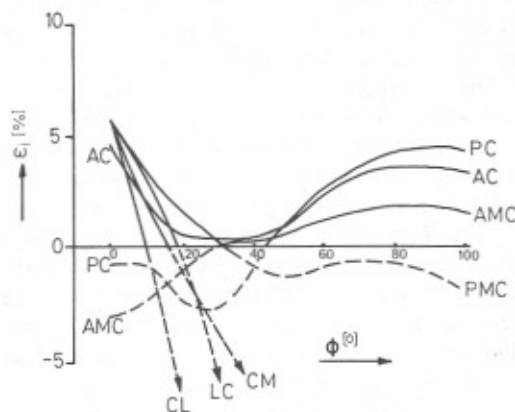


Fig. 8. Calculated strains ϵ_i as function of the flexion-extension angle ϕ .

project, so model predictions could only be compared with experiments described in the literature. Since the experimental conditions are only partially known and on account of the variability between specimens any comparison must perform be very rough. To eliminate these effects, it is planned to conduct validation experiments with specimens whose geometrical data will be used as input for the model.

The anterior-posterior laxity, which was the special object of the results presented in this paper, was compared with experiments by Markolf (1976). The shape of the model responses and the experimental curves are quite similar (Fig. 6). The deviation in magnitude which occurs for higher force levels ($F_x > 75\text{ N}$) may be caused by the absence of menisci and of vascular, muscular and tendinous structures in the model. The influence of a compressive force on the anterior-posterior laxity, the effect of cutting ligaments, the motion of the contact points on the tibial joint surfaces and the strain pattern of the collateral ligaments and the posterior capsule also accorded quite well with experiments described in the literature. It was concluded therefore that the model presented, describes many aspects of the mechanical behaviour of the knee-joint in a realistic way.

Several sensitivity analyses were carried out to study the influence of some of the parameters on the anterior-posterior laxity. From these studies it could be concluded that the model was rather insensitive to variations in the stiffness of ligaments and capsule. The strain of the springs in extension, however, which indicates the tension state of ligaments and capsule appears to have a major effect on the anterior-posterior laxity. Consequently, in the future special attention will be given to this parameter.

Table 2. Influence of some model parameters on anterior-posterior laxity ($|F_x| = 10\text{ N}$)

	Change in anterior-posterior laxity			
	$\phi = 0$ (mm)	$\phi = 30$ (mm)	$\phi = 60$ (mm)	$\phi = 90$ (mm)
1. Stiffness k_j for all springs doubled (15 → 30... etc)*	-0.2	-0.6	-0.3	-0.3
1. Initial strain ϵ_{ij} for all springs increased by 1% (5% → 6%... etc)*	-0.1	-1.9	-0.3	-0.1
3. Initial strain ϵ_{ij} for all springs decreased by 1% (5% → 4%... etc)*	+0.1	+1.5	+0.8	+0.3
4. Displacement of femoral insertion of anterior cruciate (x: +5 mm)†		+1.5	+4	+5.5
5. Displacement of femoral insertion of anterior cruciate (x: -5 mm)†		-2.2	-0.3	-0.3
6. Displacement of femoral insertion of anterior cruciate (x: +5 mm)†	0	+0.2	-0	-0.3
7. Cut lateral collateral ligament (LC)	0	0	0	0
8. Cut medial collateral ligament (MC)	0	+1	+0.4	0
9. Cut anterior cruciate (AC)	+1.0	+11.7	+6.0	‡
10. Cut posterior cruciate (PC)	0	+9.8	+20.3	+24.4
11. Cut anterior and posterior cruciate (AC + PC)	+1.0	+20.9	+28.2	‡
Anterior-posterior laxity in reference simulation.	0.4	4.5	0.9	0.7

*Refer to Table 1.

†Displacements are defined in the extension position of the joint. The initial strains were not changed.

‡Not calculated.

The primary aim of this model is to gain a better understanding of the function of the knee joint and its several structures. Besides this, the model could be applied in several fields, e.g.:

- computations of force distributions during walking and other activities;
- evaluating surgical operations such as ligament reconstructions;
- evaluating the effects of inaccurate positioning of condylar prostheses;
- evaluating diagnostic methods for ligament injuries;
- studying the injury mechanism in a knee joint.

The model has been developed for the knee-joint, but similar joints can also be analysed with the model, if geometrical data and material characteristics are available. For other types of joints the underlying theory can be generalized to develop equivalent models.

Acknowledgement - Grateful acknowledgement is made to Z.W.O. (Dutch Organisation for the Advancement of Pure Research) for their financial support in this research work.

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NOMENCLATURE

- a** position vector of origin moving co-ordinate system
- c** position vector of a point on a tibial joint surface
- $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ unit vectors in (x, y, z) -system
- $(\mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma)$ unit vectors in (α, β, γ) -system
- f** force in a spring
- \mathbf{F}_e external prescribed force
- k** constant, characterizing the stiffness of a spring
- l** distance between femoral and tibial insertion of a spring
- l_0 unstrained length of a spring
- m** number of springs
- \mathbf{M}_e external prescribed moment
- n** degree of a polynomial

\mathbf{n}	unit outward normal vector to tibial joint surface	δ	position vector a point on a femoral joint surface
\mathbf{p}	contact force	ϵ_j	strain in a spring j
\mathbf{r}	position vector of tibial insertion of a spring	ϵ_{rj}	strain in a spring j in extension
T	(3 × 3) rotation matrix	λ	unit vector indicating the direction of the moment \mathbf{M}_j
\mathbf{v}	unit vector from femoral to tibial insertion	ρ	position vector of femoral insertion of a spring
(x, y, z)	fixed co-ordinate system connected with the tibia	τ_x, τ_y	unit tangent vectors to femoral joint surface
(x, β, γ)	moving co-ordinate system connected with femur	ϕ	flexion-extension angle
		ψ, ω	angles to describe rotations of the femur.